

Generation of nonparaxial accelerating fields through mirrors.

II: Three dimensions

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Abstract: Accelerating beams are wave packets that preserve their shape while propagating along curved trajectories. In this article, we extend the ray-based treatment in Part I of this series to nonparaxial accelerating fields in three dimensions, whose intensity maxima trace circular or helical paths. We also describe a simple procedure for finding mirror shapes that convert collimated beams into fields whose intensity features trace arcs that can extend well beyond 180 degrees.

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1. Introduction

The first article in this series [1] gave a ray-based description of the so-called “nonparaxial accelerating fields” that approximately preserve their transverse intensity profile under propagation while following a curved trajectory. The treatment was limited to two dimensions. Also presented was a way for generating these fields with mirrors, so that they can bend over angles significantly beyond a semicircle. It was argued there that, given the relation of the transverse width of the main intensity maximum to the caustic’s curvature, the path traced by these fields must be one of constant curvature, i.e., a segment of a circle. In particular, circles are invariant under rotations around their center, and so are families of straight rays that form a circular caustic.

Similar ideas hold in three-dimensions. However, the monochromatic wave equation is now invariant under more transformations: in addition to the transformations described in [1], both the paraxial and nonparaxial regimes also allow rotations around the z direction. This means that there are new options for caustic shapes associated with fields that preserve their intensity profile. In addition to curves constrained to a plane (parabolas for paraxial fields, circles for nonparaxial ones), helices are also invariant under appropriate combinations of rotations and translations, therefore having constant curvature and torsion [2]. Shape-preserving waves following helical paths have been studied by many authors, following the work of Patterson and Smith [3] and Piestun and Shamir [4]. Further, fields whose features approximately follow such paths have been used in the design of two-lobed point spread functions whose orientation provides depth information [5,6], as well as for particle manipulation [7] and in the implementation of “conveyor beams” [8].

In this article, we extend the ideas presented in [1] to three dimensions, where caustics are not curves but surfaces whose features trace circles or helices. The ray-based treatment given here leads to shapes of mirrors that produce these fields, and also explains why the caustic sheets for nonparaxial fields whose maxima trace circles necessarily open away from each other more than in the paraxial case. This treatment also shows that for fields tracing helical paths the caustic sheets cannot be degenerate as they are for paraxial Airy beams and nonparaxial fields tracing circular paths.

2. Fields whose maxima trace circular paths

We now consider shape-preserving accelerating fields in three dimensions, whose intensity features trace circles. Several types of such fields have been studied recently [9–13]. In all these cases, the main intensity lobe(s) can be associated with the junction at normal angles of two sheets of a caustic surface. This junction, misleadingly referred to as a *cusp* in [9,10], is referred to here as a *ridge*. In the vicinity of this ridge, the field’s transverse intensity profile is composed of an array of secondary lobes distributed roughly over a quadrant of a distorted Cartesian grid, with the main lobe near the caustic’s ridge. This is shown in Fig. 1 for accelerated fields obtained through separation of variables in parabolic, spherical, and the two types of spheroidal coordinates [9–11]. Near the caustic ridges, these intensity profiles are similar to that of paraxial Airy beams in three dimensions (often referred to as 2D Airy beams, given their Airy function dependence in two directions), also shown in Fig. 1, where the array of intensity lobes remains straight even away from the ridge. Paraxial Airy beams correspond to rays forming a caustic structure known as a hyperbolic umbilic, which in this case happens to be degenerate [14, 15]. At any plane of constant z , the sections of the caustic sheets are straight lines intersecting at

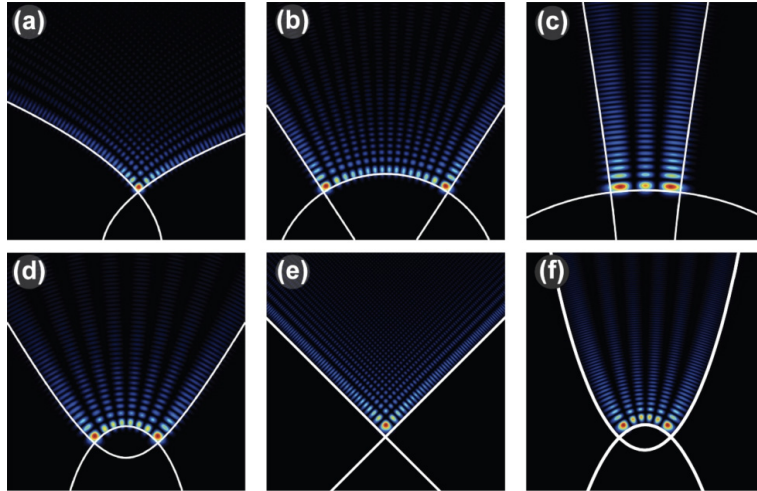


Fig. 1. Transverse cross-sections of the intensity profile and corresponding caustics (white lines) for accelerated fields whose caustic sheets are sections of (a) two confocal paraboloids, (b) a two-sheeted cone and a sphere, (c) a two-sheeted hyperboloid and a prolate spheroid, and (d) a one-sheeted hyperboloid and an oblate spheroid. Also shown in (e) are the corresponding intensity and caustic cross sections for a paraxial Airy beam and (f) paraxial accelerating parabolic beam.

the ridge at a right angle. The degeneracy of the caustic means that each of its two sections is actually the perfect superposition of two sheets, since two different rays touch each point of the caustic. It is this degeneracy that gives rise to the oscillations of the intensity along the caustic; when a limitation in the field's plane wave spectrum removes the contributions from one of the two overlaying sheets, these oscillations disappear in the corresponding region [14].

The nonparaxial accelerating fields described in [9, 10], whose intensity cross sections are shown in Fig. 1(a)–1(d), correspond to coordinate systems under which the Helmholtz equation is separable and where one coordinate is an azimuthal angle. From the ray-optical point of view, these fields also correspond to ray families that form a degenerate hyperbolic umbilic caustic [or two of them, in the case of (b-d)], with different overlaying pairs of caustic sheet shapes that intersect at a circular ridge: (a) paraboloid segments for parabolic waves, (b) cone and sphere segments for spherical waves, and (c,d) spheroid and hyperboloid segments for spheroidal waves. In all these cases, the transverse cross sections of the caustic are composed of line segments (each of them degenerate) that intersect at a ridge at a right angle, as in the paraxial regime shown in (e,f). However, unlike for the paraxial Airy beams in (e), two caustic cross sections intersecting at a ridge cannot both be straight; instead, they must open away from each other, as can be seen in Fig. 1(a)–1(d), particularly in (a). The ray-optical treatment that follows explains this fundamental difference in the behavior of paraxial and nonparaxial caustics.

3. Ray-based analysis of the caustic structure

Consider a field whose intensity profile is rotationally symmetric (at least approximately) around the y axis. Let ϕ be the azimuthal angle around the y axis, measured from the positive z axis towards the positive x axis, so that $\tan \phi = x/z$. The field then has form $U(\mathbf{r}) = u(y, \rho) \exp(im\phi)$, where $\rho = \sqrt{x^2 + z^2}$ and m is positive (and integer in the case of full rota-

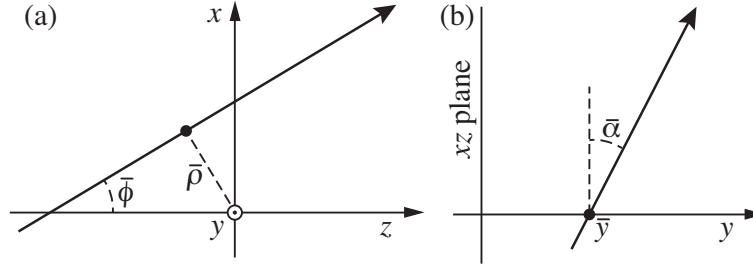


Fig. 2. (a) View of the projection of the ray onto the xz plane. (b) View of the projection onto a plane parallel to the ray and to the y axis. In both parts, the black dot indicates $\mathbf{r}_0 = (\bar{\rho} \cos \bar{\phi}, \bar{y}, -\bar{\rho} \sin \bar{\phi})$.

tional symmetry).

Consider now the ray picture. In free space rays are straight lines, each of which is fully specified by four parameters. For our purposes, it is convenient to use the parameters illustrated in Fig. 2: the angle $\bar{\phi}$ between the z axis and the projection of the ray onto the xz plane; the minimum distance $\bar{\rho}$ between the ray and the y axis; the angle $\bar{\alpha}$ between the ray and the xz plane; and $\bar{y}$, which is the y coordinate of the point at which the ray comes closest to the y axis. The ray then propagates in the direction of the unit vector $\mathbf{u} = (\sin \bar{\phi} \cos \bar{\alpha}, \sin \bar{\alpha}, \cos \bar{\phi} \cos \bar{\alpha})$, and is composed of the points $\mathbf{r} = \mathbf{r}_0 + \tau \mathbf{u}$, with $\mathbf{r}_0 = (\bar{\rho} \cos \bar{\phi}, \bar{y}, -\bar{\rho} \sin \bar{\phi})$ (shown as a black dot in Fig. 2) and where τ is a distance along the ray.

A field is associated with a two-parameter family of rays, so two constraints must be imposed involving the four ray parameters. These two constraints must be such that the resulting two-parameter family of rays constitutes a *normal congruence*, i.e., that all rays intersect perpendicularly a family of surfaces. (Sufficiently far way from caustics, these surfaces nearly coincide with wavefronts; at caustics, on the other hand, these surfaces fold.) It is clear that the two required constraints cannot involve $\bar{\phi}$, since like the field, the ray family must have rotational symmetry around the y axis. That is, $\bar{\phi}$ is one of the two parameters needed to label each ray, and $\bar{\rho}$, $\bar{\alpha}$ and $\bar{y}$ are independent of it.

The first constraint results from the rotational symmetry around the y axis; the angular momentum's y component for each ray must be m , a constant, i.e.,

$$[\mathbf{r} \times (k\mathbf{u})]_y = k\bar{\rho} \cos \bar{\alpha} = m, \quad (1)$$

where $\mathbf{r}$ is any point along the ray (as given by the expression above) and k is the wavenumber. That is, $\bar{\rho}$ can be considered as a function of $\bar{\alpha}$ given by $\bar{\rho}(\bar{\alpha}) = m \sec(\bar{\alpha})/k$. Similarly, $\bar{y}$ also depends on $\bar{\alpha}$ and is written for convenience here as $\bar{y}(\sin \bar{\alpha})$. However, before finding this function, we show that the condition in Eq. (1) alone guarantees that the resulting ray family constitutes a normal congruence. First, note that an infinitesimal variation $\delta\bar{\phi}$ in $\bar{\phi}$ causes an infinitesimal rotation in Fig. 2(a), which corresponds to an infinitesimal vertical displacement of the ray in Fig. 2(b) (including the black dot representing $\mathbf{r}_0$) of magnitude $-\bar{\rho}\delta\bar{\phi}$. This vertical displacement introduces an optical path difference of magnitude $\delta\Phi = \cos \bar{\alpha} \bar{\rho} \delta\bar{\phi}$. By using Eq. (1), it is easy to see that the change of optical path length with respect to $\bar{\phi}$ is then given by

$$\frac{\partial \Phi(\bar{\alpha}, \bar{\phi})}{\partial \bar{\phi}} = \frac{m}{k}. \quad (2)$$

On the other hand, as can be seen from Fig. 2(b), a small change $\delta\bar{\alpha}$ in $\bar{\alpha}$ causes an infinitesimal rotation of the ray, which causes a change in optical path length $\delta\Phi = -\bar{y} \cos \bar{\alpha} \delta\bar{\alpha}$, so the

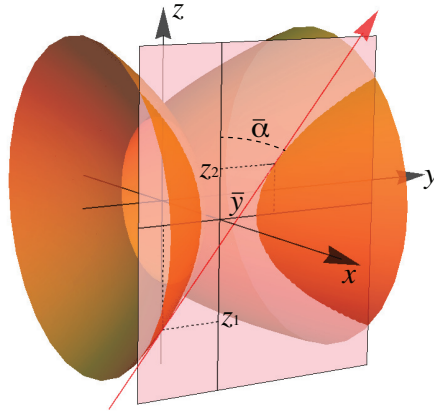


Fig. 3. View of the two caustic sheets and a ray with $\bar{\phi} = 0$ and given $\bar{\alpha}$.

change of Φ with respect to $\bar{\alpha}$ is

$$\frac{\partial \Phi(\bar{\alpha}, \bar{\phi})}{\partial \bar{\alpha}} = -\bar{y} \cos \bar{\alpha} \quad (3)$$

One can show that the following form for Φ satisfies the previous two equations:

$$\Phi(\bar{\alpha}, \bar{\phi}) = \frac{m}{k} \bar{\phi} - \bar{Y}(\sin \bar{\alpha}), \quad \bar{Y}(u) = \int^u \bar{y}(u') du', \quad (4)$$

so the ray family constitutes a normal congruence. If one wants to calculate the wave field associated with this family of rays, its angular spectrum is given by

$$A(\bar{\alpha}, \bar{\phi}) = \mathcal{A}(\bar{\alpha}, \bar{\phi}) \exp[ik\Phi(\bar{\alpha}, \bar{\phi})], \quad (5)$$

where $\mathcal{A}$ is a real amplitude factor. Since $(\bar{\alpha}, \bar{\phi})$ are spherical angular coordinates around the y axis, where $\bar{\alpha}$ is measured from the equatorial xz plane, the Jacobian metric needed to integrate over the sphere is $\cos \bar{\alpha}$.

From Eq. (1), the radius of the caustic's ridge, given by the smallest value of $\bar{\rho}$, corresponding to $\bar{\alpha} = 0$, is

$$\bar{\rho}_0 = \frac{m}{k}. \quad (6)$$

The geometric interpretation of this result is simple: the ridge's perimeter fits m wavelengths. For $\bar{\rho} > \bar{\rho}_0$, the caustic could open into four sheets with rotational symmetry around the y axis. However, we choose to make the caustic sheets degenerate so that there are only two of them, as shown in Fig. 3. Each ray touches each of the two sheets, so that each point in the caustic (except for the ridge) is touched by two rays, corresponding to opposite signs of $\bar{\alpha}$ and to two different values of $\bar{\phi}$. As with paraxial Airy beams, the fact that each point in the sheet is touched by two rays is what makes the caustic sheets degenerate and what gives rise to intensity oscillations over these sheets.

The caustic sheets cannot have arbitrary shapes, as we now show. Let the two sheets be given by the equations $y = h_n(\rho)$ for $n = 1, 2$, where $\rho \geq \bar{\rho}_0$ is the distance from the y axis, with

the constraint $h_n(\bar{\rho}_0) = \bar{y}(0)$. Consider a ray with fixed $\bar{\alpha}$, and let $\bar{\phi} = 0$ for simplicity. Since $\bar{\alpha}$ is fixed, so is $\bar{\rho}(\bar{\alpha})$ which equals a constant $x_0 > \bar{\rho}_0$. The ray is contained within the plane $x = x_0$ (shown as a pink rectangle in Fig. 3). This plane intersects both caustic sheets at two curves with equations $y = h_{1,2}[(x_0^2 + z^2)^{1/2}]$. Since the ray touches tangentially the two curves, the following two sets of conditions must be satisfied:

$$\tan \bar{\alpha} = \frac{h_n(\sqrt{x_0^2 + z_n^2}) - \bar{y}}{z_n}, \quad \tan \bar{\alpha} = \left. \frac{d h_n(\sqrt{x_0^2 + z^2})}{d z} \right|_{z=z_n}, \quad (7)$$

for $n = 1, 2$, where z_n are the z coordinates of the two points at which the ray touches the caustic sheets. Substituting $\rho_n = \sqrt{x_0^2 + z_n^2}$, these equations lead to

$$\tan \bar{\alpha} = \frac{h_n(\rho_n) - \bar{y}}{\sqrt{\rho_n^2 - x_0^2}}, \quad \tan \bar{\alpha} = \frac{\sqrt{\rho_n^2 - x_0^2}}{\rho_n} h'_n(\rho_n). \quad (8)$$

On the other hand, Eqs. (1) and (6) imply that $\cos \bar{\alpha} = \bar{\rho}_0/x_0$, and therefore we can rewrite these equations as

$$h_n(\rho_n) = \bar{y}(\sin \bar{\alpha}) + \tan \bar{\alpha} \sqrt{\rho_n^2 - \bar{\rho}_0^2} \sec^2 \bar{\alpha}, \quad (9)$$

$$h'_n(\rho_n) = \frac{\rho_n \tan \bar{\alpha}}{\sqrt{\rho_n^2 - \bar{\rho}_0^2} \sec^2 \bar{\alpha}}. \quad (10)$$

Equations (9) and (10) rule the behavior of the caustic sheets. Suppose that h_1 is given. Then Eq. (10) with $n = 1$ can be solved for ρ_1 , and this result can be substituted in Eq. (9) to find $\bar{y}(\sin \bar{\alpha})$. Note that this functional form of $\bar{y}$ can be used in Eq. (4) to determine Φ . This result for $\bar{y}$ can then also be substituted in Eq. (9) for $n = 2$, which is then solved for $\tan \bar{\alpha}$ and the solution substituted into Eq. (10), leading to a nonlinear differential equation for $h_2(\rho_2)$.

In general, the procedure just outlined must be performed numerically. For the cases of the caustic shapes in Fig. 1(a)–1(d), however, this system of equations has simple solutions. For example, one can show that one set of solutions is given by confocal paraboloids of the form:

$$h_{1,2}(\rho) = \mp e^{\mp a} \frac{\rho^2 - \bar{\rho}_0^2}{2\bar{\rho}_0}, \quad \bar{y}(\sin \bar{\alpha}) = \bar{\rho}_0 \sinh a \tan^2 \bar{\alpha}. \quad (11)$$

The slopes with respect to the y axis of the sheets at the caustic's ridge are $-e^a$ and e^{-a} , respectively, whose product gives -1 , consistently with the fact that they intersect at a right angle. Further, the resulting phase of the angular spectrum is given according to Eq. (4) by

$$k\Phi = k\bar{\rho}_0 \sinh a \left[\sin \bar{\alpha} - \ln \left(\frac{1 + \sin \bar{\alpha}}{\cos \bar{\alpha}} \right) \right] + m\bar{\phi}, \quad (12)$$

which coincides exactly with that for the separable symmetric paraboloidal waves in [10] (after an appropriate shift in y), shown in Fig. 1(a).

Equations (9) and (10) can also be solved exactly for the other cases associated with separation of variables in [9] and [10]: if one caustic sheet is a cone, the second is a segment of a sphere (and not another cone, explaining why the cross sections of the caustics cannot both be straight); if one is a sheet from a two-sheeted hyperboloid, the second is a prolate spheroid; if

the first is a one-sheeted hyperboloid, the second is an oblate spheroid. For these cases, however, using an angular spectrum whose phase is k times the ray-based Φ in Eq. (4) leads to a field with the desired caustic shapes but that presents only one ridge, therefore not matching globally the corresponding fields that result from separation of variables described in [9] and [10], which present two ridges.

We now show that, as mentioned earlier, the two caustic sheets must intersect at the ridge at right angles, regardless of their shape away from it. For this purpose, we consider only rays that touch the caustic near the ridge, which correspond to $|\bar{\alpha}| \ll 1$. We also introduce $\Delta_n = \rho_n - \bar{\rho}_0$ and assume $\Delta_n \ll \bar{\rho}_0$. We choose for simplicity $\bar{y}(0) = h_n(\bar{\rho}_0) = 0$, and notice that the degeneracy of the caustic requires $\bar{y}$ to be an even function. The series expansions of these functions can then be written as $\bar{y}(\sin \bar{\alpha}) = \bar{y}''(0)\bar{\alpha}^2/2 + \mathcal{O}(\bar{\alpha}^4)$ and $h_n(\rho_n) = h'_n(\bar{\rho}_0)\Delta_n + \mathcal{O}(\Delta_n^2)$, and their substitution in Eqs. (9,10) leads to

$$h'_n(\bar{\rho}_0)\Delta_n \approx \bar{y}''(0)\frac{\bar{\alpha}^2}{2} + \bar{\alpha}\sqrt{2\bar{\rho}_0\Delta_n - \bar{\rho}_0^2\bar{\alpha}^2}, \quad (13)$$

$$h'_n(\bar{\rho}_0)\sqrt{2\bar{\rho}_0\Delta_n - \bar{\rho}_0^2\bar{\alpha}^2} \approx (\bar{\rho}_0 + \Delta_n)\bar{\alpha}, \quad (14)$$

where we only kept terms up to linear in Δ_n and quadratic in $\bar{\alpha}$. The solution for Δ_n of Eq. (14) gives $\Delta_n = \{[h'_n(\bar{\rho}_0)]^{-2} + 1\}\bar{\rho}_0\bar{\alpha}^2/2 + \mathcal{O}(\bar{\alpha}^4)$, which substituted into Eq. (13) leads to a quadratic equation for $h'_n(\bar{\rho}_0)$ whose solutions are

$$h'_{1,2}(\bar{\rho}_0) = \frac{\bar{y}''(0)}{2\bar{\rho}_0} \pm \sqrt{\left[\frac{\bar{y}''(0)}{2\bar{\rho}_0}\right]^2 + 1}. \quad (15)$$

That is, the slopes of the sheets at the ridge are determined by $\bar{y}''(0)$. In particular, notice that $h'_1(\bar{\rho}_0)h'_2(\bar{\rho}_0) = -1$, which confirms that the caustic sheets intersect at the ridge at a right angle.

4. Mirrors that generate the prescribed fields

As in the two-dimensional case in [1], we now discuss the shape that a curved mirror must have to generate fields with the desired caustic shape and whose main intensity features trace more than a semicircle. We use an optical path length argument analogous to the string picture used in [1]. For simplicity we consider only incident collimated fields, but we allow them to arrive from an arbitrary direction. The idea is to ensure that the length of a collimated ray with incident unit direction $\mathbf{n}$, from a given initial flat wavefront to a point $\mathbf{R}$ at the mirror, plus the length of the reflected ray with direction $\mathbf{u} = (\sin \phi \cos \bar{\alpha}, \sin \bar{\alpha}, \cos \phi \cos \bar{\alpha})$, from $\mathbf{R}$ to a flat wavefront containing the origin, equals $\Phi(\bar{\alpha}, \bar{\phi})$ for the desired caustic. This can be written as

$$\mathbf{n} \cdot \mathbf{R} - \mathbf{u} \cdot \mathbf{R} = T + \Phi(\bar{\alpha}, \bar{\phi}), \quad (16)$$

where, as before, $\Phi(\bar{\alpha}, \bar{\phi}) = \bar{\rho}_0\bar{\phi} - \bar{y}(\bar{\alpha})\sin \bar{\alpha}$, and T is a parameter that regulates the size of the mirror. (The minus sign in the second term of the left-hand side is because the distance is from $\mathbf{R}$ and not to $\mathbf{R}$ as in the first term.) Since the point at the mirror is along a ray segment that touches the caustic, it can be written as $\mathbf{R} = \mathbf{r}_0 + \tau\mathbf{u}$, with $\mathbf{r}_0 = (\bar{\rho} \cos \phi, \bar{y}, -\bar{\rho} \sin \phi) = (\bar{\rho}_0 \sec \bar{\alpha} \cos \bar{\phi}, \bar{y}, -\bar{\rho}_0 \sec \bar{\alpha} \sin \bar{\phi})$. The substitution of these expressions in Eq. (16) can be solved for the displacement along the ray, τ , as

$$\tau(\bar{\alpha}, \bar{\phi}) = \frac{\mathbf{n} \cdot \mathbf{r}_0 - \mathbf{u} \cdot \mathbf{r}_0 - T - \Phi(\bar{\alpha}, \bar{\phi})}{(1 - \mathbf{n} \cdot \mathbf{u})} = \frac{\mathbf{n} \cdot \mathbf{r}_0 - T - \bar{\rho}_0\bar{\phi} + \bar{y}(\sin \bar{\alpha}) - \bar{y}\sin \bar{\alpha}}{(1 - \mathbf{n} \cdot \mathbf{u})}. \quad (17)$$

The parametric equation for the reflector is then

$$\mathbf{R}(\bar{\alpha}, \bar{\phi}) = \mathbf{r}_0(\bar{\alpha}, \bar{\phi}) + \tau(\bar{\alpha}, \bar{\phi})\mathbf{u}(\bar{\alpha}, \bar{\phi}). \quad (18)$$

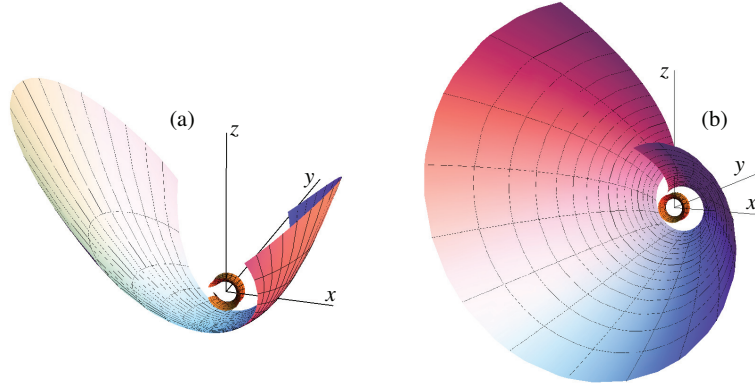


Fig. 4. Mirrors that generate symmetric paraboloidal accelerated fields and the resulting caustic segment (shown in orange) following illumination by a collimated beam that is (a) normal and (b) parallel to the caustic's axis of symmetry, for $T = 6.2R$, $-\pi/4 \leq \bar{\alpha} \leq \pi/4$ and (a) $-3\pi/4 \leq \bar{\phi} \leq 3\pi/4$, (b) $-\pi \leq \bar{\phi} \leq \pi$.

Figure 4 shows segments of two such mirrors, as well as the corresponding sections of the caustic generated by them, for the case of symmetric paraboloidal caustic sheets ($\bar{y} = 0$) when the illuminating light comes from the top [the source is at $(0, 0, \infty)$] and from the side [the source is at $(0, -\infty, 0)$], both for $T = 6.2R$. In the first case, the $y = 0$ section of this mirror corresponds to the shape of the mirror found for the two-dimensional case in [1], while in the second it corresponds to a spiral given by the involute of the caustic circle over its plane. The second configuration presents several advantages. Firstly, following a possible second reflection, the mirror naturally steers the rays away from the caustic. Secondly, interference of the caustic with the incident field can be suppressed through placement of a suitable circular obscuration in front of the mirror. Thirdly, the caustic can be made to trace essentially a complete circle without the need for the mirror to be too large.

As in the two-dimensional case in [1], it is easy to find an apodization of the incident field that makes the intensity along the caustic as uniform as possible by calculating the Jacobian of the two mirror coordinates transverse to the intensity field in terms of the two angles. Also, one can consider not only monochromatic fields but also pulsed ones, leading to localized intensity maxima that trace paths that are arbitrarily close to a full circle.

5. Helical accelerated waves

As mentioned in the introduction, one can also define fields that preserve their transverse profile while tracing a helical path. We can adapt the ray-based treatment given earlier to turn the ridge of the caustic into a helix forming an angle γ with the y axis. To do this, we first add linear dependence on $\bar{\phi}$ to the lateral displacement $\bar{y}$. Notice, though, that transforming the circular path into a helical one means that the ray touching the ridge no longer corresponds to $\bar{\alpha} = 0$ but to $\bar{\alpha} = \pi/2 - \gamma$. Therefore, we replace $\bar{Y}$ with $\bar{Y}_\gamma$, given by

$$\bar{Y}_\gamma(\sin \bar{\alpha}, \bar{\phi}) = \bar{Y}[-\cos(\bar{\alpha} + \gamma)] + \bar{\rho}_0 \bar{\phi} \cot \gamma \sin \bar{\alpha}. \quad (19)$$

Also, the $\bar{\phi}$ dependent part of Φ , namely $\bar{\rho}_0 \bar{\phi}$, must be replaced with $\bar{\rho}_0 \bar{\phi} \csc \gamma$, since the path length per unit angle $\bar{\phi}$ is no longer $\bar{\rho}_0$ but $\bar{\rho}_0 \csc \gamma$. If this replacement were not made, the

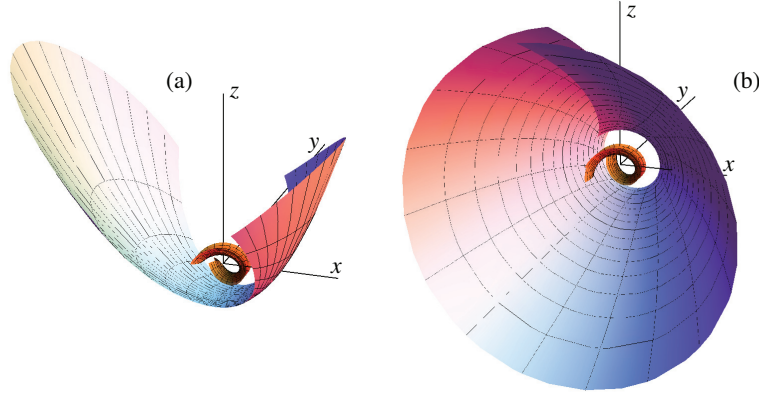


Fig. 5. (a,b) Mirrors that generate helical accelerated fields and the resulting caustic segment (shown in orange) following illumination by a collimated beam that is (a) normal and (b) parallel to the caustic's axis of symmetry, for $T = 6.2R$, $-\pi/4 \leq \bar{\alpha} \leq \pi/4$ and (a) $-3\pi/4 \leq \bar{\phi} \leq 3\pi/4$, (b) $-\pi \leq \bar{\phi} \leq \pi$.

radius of the helical ridge would shrink, just as the radius of a spring shrinks when we stretch it. Following these two replacements, instead of $k\Phi$, the phase of the angular spectrum is $k\Phi_\gamma$, where

$$\Phi_\gamma(\bar{\alpha}, \bar{\phi}) = \bar{\rho}_0 \bar{\phi} \frac{1 - \sin \bar{\alpha} \cos \gamma}{\sin \gamma} - \bar{Y}[-\cos(\bar{\alpha} + \gamma)]. \quad (20)$$

(Clearly, the case of fields whose intensity features follow circular paths is recovered for $\gamma = \pi/2$.) Notice that the constraint in the ray parameters $\bar{\rho}(\bar{\alpha}) = \bar{\rho}_0 / \cos \bar{\alpha}$ of fields following circular paths generalizes for the helical case as

$$\bar{\rho}(\bar{\alpha}) = \frac{1 - \sin \bar{\alpha} \cos \gamma}{\sin \gamma \cos \bar{\alpha}} \bar{\rho}_0. \quad (21)$$

The field (in the Debye approximation [16] where a plane wave is assigned to each ray) is given, in cylindrical coordinates $\mathbf{r} = (\rho \sin \phi, y, \rho \cos \phi)$, by

$$\begin{aligned} U(\mathbf{r}) &= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \mathcal{A}(\bar{\alpha}, \bar{\phi}) \exp\{ik[\Phi_\gamma(\bar{\alpha}, \bar{\phi}) + \mathbf{u} \cdot \mathbf{r}]\} \cos \bar{\alpha} d\bar{\alpha} d\bar{\phi} \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \mathcal{A}(\bar{\alpha}, \bar{\phi}) \exp\left\{ik\left[\bar{\rho}_0 \bar{\phi} \frac{1 - \sin \bar{\alpha} \cos \gamma}{\sin \gamma} + \rho \cos(\bar{\phi} - \phi) \cos \bar{\alpha}\right]\right\} d\bar{\phi} \\ &\quad \times \exp\{ik[y \sin \bar{\alpha} - \bar{Y}[-\cos(\bar{\alpha} + \gamma)]]\} \cos \bar{\alpha} d\bar{\alpha}. \end{aligned} \quad (22)$$

The equations for mirrors that generate such a field correspond simply to the substitution of Φ_γ instead of Φ in Eqs. (16-18). These mirrors and the corresponding caustic segments are shown, for $\gamma = \pi/4$, in Fig. 5 for the same parameters and illumination directions as in Fig. 4.

Equation (22) describes a field whose main intensity features trace a fraction of a helical turn. The turn would be complete if the magnitude of the angular spectrum, $\mathcal{A}$, were not localized to only a region within the integration interval in $\bar{\phi}$. For the field's features to trace more than one

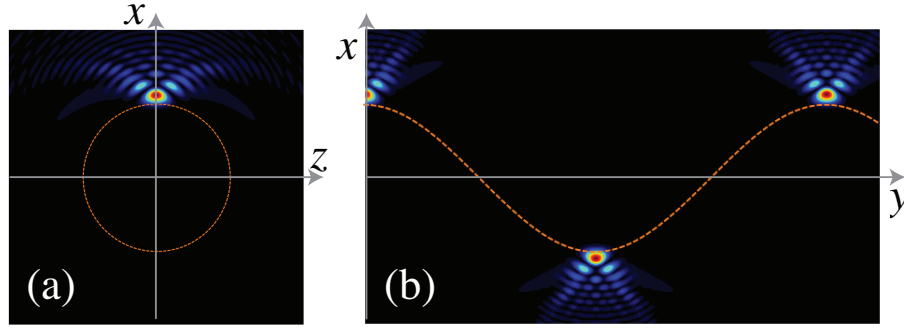


Fig. 6. (a) Transverse and (b) longitudinal intensity patterns for a helical version of a symmetric paraboloidal field ($a = 0$) for $k\bar{\rho}_0 = 50$ and $\gamma = \pi/4$. The dashed orange circle in (a) and sinusoidal in (b) show the projections onto the corresponding planes of the helical path followed by the caustic's ridge. Note that both figures contain the same information, and that varying y would simply cause the transverse profile in (a) to rotate, while varying ϕ would cause the radial section in (b) to move laterally.

turn, one can conceptually join several of these fields using different windows of integration in $\bar{\phi}$, i.e., extending the integral in $\bar{\phi}$ over more than one cycle. In fact, if we assume that $\mathcal{A}$ is independent of $\bar{\phi}$, we can let the region of integration in $\bar{\phi}$ grow to all reals while renormalizing $\mathcal{A}$. In the limit, the integral differs from zero only when $\bar{\rho}_0\bar{\phi}(1 - \sin\bar{\alpha}\cos\gamma)\csc\gamma$ is an integer. That is, the only contributions are for $\bar{\alpha} = \bar{\alpha}_n$, where n is an integer, given by

$$\bar{\alpha}_n = \arcsin \left[\left(1 - \frac{n \sin \gamma}{k\bar{\rho}_0} \right) \sec \gamma \right]. \quad (23)$$

For these values of $\bar{\alpha}$ the (renormalized) integral in $\bar{\phi}$ gives Bessel functions of the first kind, J_n , so the field is given by

$$U_H(\mathbf{r}) = \sum_{n=n_{\min}}^{n_{\max}} \mathcal{A}(\bar{\alpha}_n) \cos^2 \bar{\alpha}_n J_n(k\rho \cos \bar{\alpha}_n) \times \exp[i(n\phi + \{y \sin \bar{\alpha}_n - \bar{Y}[-\cos(\bar{\alpha} + \gamma)]\})], \quad (24)$$

where $n_{\min, \max} = \mp[k\bar{\rho}_0(1 \mp \cos\gamma)\csc\gamma]$, and the second factor of $\cos\bar{\alpha}_n$ comes from the derivative of the argument of the delta function resulting from the integral in $\bar{\phi}$. This type of superposition of Bessel beams has been shown [4] to lead to fields whose features follow helical paths, although the superposition in Eq. (24) involves counter-propagating Bessel beams as well. Cross sections of the intensity are shown in Fig. 6 for the case with $m = 50$, $\bar{y} = 0$ and $\gamma = \pi/4$.

It must be noted that helical fields whose maxima describe more than one cycle can no longer be generated by a single unobstructed mirror transforming light from a single source (unless perhaps if the mirror is partially transmissive). However, the idealized fields given by the expression in Eq. (24) have locally the same connection to the ray structure as those that can be generated by a mirror, while simplifying the evaluation of the field. This allows us to explore the effects on the transverse intensity profile of making the field follow a helical path, and to compare them to the corresponding effects on the ray family. For simplicity, we consider the case of helical versions of fields whose caustic cross-sections are parabolic, so that $\bar{y}(u) = \bar{\rho}_0 \sinh au^2/(1 - u^2)$ following Eq. (11). Figures 7(a)–7(d) show, for the case $k\bar{\rho}_0 = m = 50$,

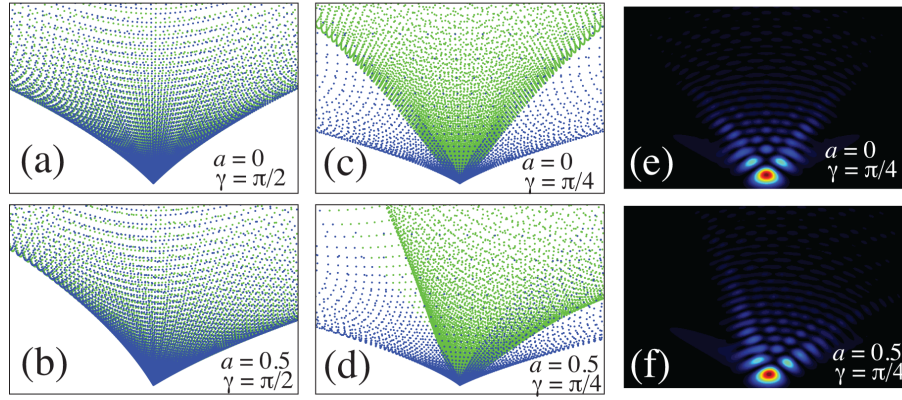


Fig. 7. (a-d) Ray and (e,f) intensity cross sections over the $z = 0$ plane within $-80 \leq ky \leq 80$ and $48 \leq kx \leq 150$, for fields with $\bar{y}(u) = \bar{\rho}_0 \sinh au^2 / (1 - u^2)$ for $k\bar{\rho}_0 = 50$ and different values of γ and a . In (a-d), blue (green) dots correspond to rays for which $\bar{\alpha} > (<) \pi/2 - \gamma$.

cross sections of a set of sample rays over a segment of the $z = 0$ plane corresponding to $-80 \leq ky \leq 80$ and $48 \leq kx \leq 150$, first for the circular (non-helical) case $\gamma = \pi/2$ with (a) no tilt ($a = 0$) and (b) some tilt corresponding to $a = 0.5$. The same two cases are shown in (c) and (d) when the field is helical with $\gamma = \pi/4$. These figures show that the helicity of the path causes the caustic structure to no longer be degenerate, that is, we see four separate caustic sheets. The caustics due to the rays for which $\bar{\alpha} > \pi/2 - \gamma$ (i.e., that form an angle with the y axis smaller than that of the helix), shown as blue dots, are more open than those for the rays for which $\bar{\alpha} < \pi/2 - \gamma$ (i.e., that form an angle with the y axis that is larger than that of the helix), shown as green dots. The behavior of the caustics in (c) and (d) are reflected by the corresponding intensity cross sections in (e) and (f), respectively. Notice that the intensity at regions corresponding to the outermost caustic branches no longer show intensity oscillations, given that there are no longer two superposed caustic contributions interfering.

The fact that caustic sheet degeneracy does not hold for helical fields follows from the corresponding generalizations of Eqs. (9,10) that rule the caustics $y = h_n$, resulting from replacing $\bar{\rho}_0 \sec \bar{\alpha}$ with the expression in Eq. (21) and adding a term proportional to ϕ the first equation:

$$h_n(\rho_n) = \frac{\sin(\bar{\alpha} + \gamma)}{\cos \bar{\alpha}} \bar{y} [-\cos(\bar{\alpha} + \gamma)] + \tan \bar{\alpha} \sqrt{\rho_n^2 - \kappa^2 \bar{\rho}_0^2} + \bar{\rho}_0 \cot \gamma \arctan \left[\frac{\sqrt{\rho_n^2 - \kappa^2 \bar{\rho}_0^2}}{\kappa \bar{\rho}_0} \right], \quad (25)$$

$$h'_n(\rho_n) = \frac{\rho_n^2 \tan \bar{\alpha} - \kappa \bar{\rho}_0^2 \cot \gamma}{\rho_n \sqrt{\rho_n^2 - \kappa^2 \bar{\rho}_0^2}}, \quad (26)$$

where $\kappa = (1 - \sin \bar{\alpha} \cos \gamma) / (\sin \gamma \cos \bar{\alpha})$. One can see from these equations that, for a given $\bar{y}$, the solutions for h_n for $\bar{\alpha} < \pi/2 - \gamma$ differ from those for $\bar{\alpha} > \pi/2 - \gamma$. Through an analysis similar to that at the end of Section 3 where only rays for which $\bar{\alpha} \approx \pi/2 - \gamma$ (i.e., that touch the caustic near the ridge) are considered, one can find that $h'_1(\bar{\rho}_0)h'_2(\bar{\rho}_0) = -\csc^2 \gamma$, which means that both sets of sheets still intersect normally at the ridge; the factor of $\csc^2 \gamma$ on the right accounts for the obliquity due to the angle between the ridge and the plane of constant ϕ where h_n is defined. The two sets of sheets locally overlap at the ridge, but then bend away

from each other.

6. Concluding remarks

The ray treatment presented in this article had two main objectives. The first was to show that three-dimensional fields whose transverse intensity profile is largely preserved under propagation while the field describes a curved (circular or helical) path spanning an angle arbitrarily close to 2π can be generated by suitably shaped mirrors. The second was to gain a better understanding of the three-dimensional caustic structure of these fields. It was found that, for fields following circular paths, there is some freedom on the choice of the shape of one of the (degenerate) caustic sheets, but that then the second sheet is fully determined through a nonlinear differential equation. In particular, this treatment shows that it is not possible to create a nonparaxial accelerated field following a circular path where the caustic sheets are both cones, so that its profile looks like the one of paraxial Airy beams. For the cases corresponding to separable coordinate systems of the Helmholtz equation with axial symmetry, the pair of caustic sheets do coincide with surfaces of the coordinate system in question.

When the fields are made to follow helical paths, the degeneracy of the caustics is broken. Mathematically, this phenomenon is analogous to many others in physics, where the introduction of a perturbation splits degenerate levels of a physical quantity, as in spin-orbit interaction in quantum mechanics or in nonparaxial optics. In particular, the subfamilies of rays forming angles with the axis of rotation that are smaller or larger than that of the caustic's ridge give rise to caustic sheets intersecting at different angles. One could engineer a field where one of the two caustic sheets for each subfamily are forced to coincide, but these would exacerbate the difference of the remaining two sheets.

As a final comment, note that the treatment in this article is scalar, but it can be extended to electromagnetic fields easily by using the solutions discussed here as Hertz potentials in an electromagnetic treatment, as in [10].

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